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# THE APPROXIMATION OF REACHABLE DOMAINS OF CONTROL SYSTEMS†

Kh. G. GUSEINOV, A. N. MOISEYEV and V. N. USHAKOV

Ekaterinburg

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The problem of constructing reachable domains (RDs) of a non-linear control system functioning over a finite time interval is considered. A method is proposed for the approximate construction of RDs, based on partitioning the phase space of the system by an ε-lattice. Estimates are obtained for the accuracy of the approximate RDs. An example is presented. © 1998 Elsevier Science Ltd. All rights reserved.

The numerous publications devoted to this problem present a variety of approaches to its solution. A large group of publications [1–5] is devoted to estimating reachable domains (RDs) of control systems and differential inclusions (DIs) by ellipsoids or sets of ellipsoids in the space $R^n$. Estimates of RDs have been obtained [6, 7]. The problem of the approximate computation of RDs as polyhedra in the phase space $R^m$ has been investigated [8]. Mention should also be made of papers [9, 10] on numerical methods for constructing RDs of linear control system.

The approach adopted here is close to that of [11, 12], which was based on introducing a rectangular lattice in the position space and approximating RDs by sets of lattice nodes. A similar method was used in [13–15]. The investigations in [16], which consider the approximate computation of RDs in the case of autonomous DIs, are more recent. Other properties of RDs have also been investigated [17–24].

**1.** Suppose we are given a control system whose behaviour is described by the equation

$$\dot{x}=f(t,\ x,\ u),\quad u\in P,\quad t\in I,\quad I=[t_0,\ \vartheta],\quad t_0<\vartheta<\infty \tag{1.1}$$

where $x$ is the $m$-dimensional vector of the system, $u$ is the control and $P$ is a compact subset of Euclidean space $R^n$.

It is assumed that the following conditions are satisfied

1. The vector function $f(t, x, u)$ is continuous in the set of variables $t, x, u$ in the domain $I \times R^m \times P$, and for any bounded and closed domain $D \subset I \times R^m$ a constant $L = L(D) \in (0, \infty)$ exists such that

$$\| f(t,x^*,u)-f(t,x_*,u)\| < L\| x^*-x_*\| \text{ for } (t,x^*) \text{ and } (t,x^*) \text{ in } D,\ u,\in P.$$

2. A constant $u \in [0, \infty)$ exists such that

$$\|f(t,\ x,\ u)\| \le \mu(1+\|x\|),\quad (t,\ x,\ u)\in D\times P$$

By an admissible control $u(t)$, $t \in I$, we mean any Lebesgue-measurable function such that $u(t), \in P$, $t \in I$.

A solution of Eq. (1.1) generated by an admissible control $u(t)$ is defined as an absolutely continuous vector function $\dot{x}[t]$, $t \in I$, such that $\dot{x}[t] = f(t, x[t], u(t))$ almost everywhere (a.e.) on $I$.

The symbol $Y(t^*; t_*, x_*)$, $t_0 \leqslant t_* < t^* \leqslant \vartheta$ will denote the set of all $x^* \in R^m$ reached at time $t^*$ by solutions $x[t], x[t_*] = x_*$ of Eq. (1.1) generated by all possible admissible controls $u(t)$; $Y(t^*; t_*, x_*)$ is called the RD at time $t^*$ of system (1.1) with initial data $x[t_*] = x_*$. We also assume

$$Y(t^*;\ t_*,Y_*)=\bigcup_{x_*\in Y_*}Y(t^*;\ t_*,\ x_*)$$

($Y_*$ is a set from $R^m$).

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We will associate a DI

$$\dot{x} \in F(t,x),\ t \in I,\ F(t,x) = \mathrm{co}\{f(t,x,u) : u \in P\} \tag{1.2}$$

where co{•} denotes the closed convex hull of the set in question, with Eq. (1.1).

A solution of the DI (1.2) is defined as an absolutely continuous vector function $x[t]$, $t \in I$ satisfying (1.2) a.e. on $I$.

Let $X_*^* = X(t^*; t_*, X_*)$, $t_0 \leq t_* < t^* \leq \vartheta$ denote the set of all $x^* \in R^m$ reached at time $t^*$ by all possible solutions $x[t], x[t_*] = x_* \in X_*$ of the DI (1.2). We have

$$X_*^* = \mathrm{cl}Y(t^*;\ t_*,\ X_*),\ t_0 \leq t_* < t^* \leq \vartheta,\quad X_* \subset R^m$$

where cl$X$ denotes the closure of $X$.

Thus, let us consider the problem of computing the set $X_0^* = X(t^*; t_0, X_0)$, $t^* \in I, X_0 \subset R^m$, frequently encountered in theory and applications. In the general case this set cannot be calculated exactly, and we will therefore have to do so approximately.

The right-hand side $F(t,x)$ of the DI (1.2) generally depends on $t$ and satisfies a Lipschitz condition

$$d(F(t,x), F(t,y)) \leq L\|x-y\| \text{ for } (t,x), (t,y) \text{ in } D, L = L(D) \in [0,\infty),$$

where $D \subset I \times R^m$ is any bounded closed domain.

Note that if the set $X_0$ is bounded, it follows from Condition 2 that the sets $X_0^*$, $t^* \in I$ are uniformly bounded. It is not difficult to show, using Gronwall's inequality (see, for example, [25]) that a cylinder $D = I \times G$ in the space of positions $(t,x)$ exists such that $X_0^* + \varepsilon^* B \subset G$, $t^* \in I$ (where $G$ is a closed sphere in $R^m$, $B = \{x \in R^m : \|x\| \leq 1\}$, $\varepsilon^* > 0$), where we have used the notation $X + Y = \{x + y : x \in X, y \in Y\}$ $\alpha X = \{\alpha x : x \in X\}$.

On the basis of this remark, we shall assume that all our constructions are carried out in the cylinder $D$ and therefore, for all $(t,x)$ occurring in these constructions, the right-hand sides $F(t,x)$ of the DI (1.2) are uniformly bounded (that is, $\|f\| \leq K$ for all $f \in F(t,x)$, $(t,x) \in D$, where $K$ is some finite positive number).

The approach proposed here is based on introducing a certain finite partition $\Gamma$ of the interval $I$, substituting for the phase space $R^m$ a certain fixed ε-lattice $N_\varepsilon$ and approximating the RDs $X_0^*$, $t^* \in I$ by certain finite subsets of $N_\varepsilon$. The computation of the RDs $X_0^*$, $t^* \in I$ of the DI (1.2) reduces to a certain procedure of calculating discrete approximations of these domains—finite subsets of the lattice $N_\varepsilon$.

**2.** We now introduce a set $\tilde{X}_1 = \tilde{X}(t_1; t_0, X_0)$ that will serve as an approximation of the RD $X_1 = X(t_1; t_0, X_0)$ and which corresponds to the DI $\dot{x}(t) \in F(t_0, x_0)$; we will then determine the degree to which the set $X_1$ is approximated by $\tilde{X}_1$.

Let $\Gamma$ be a partition of the interval $I$ by times $t_0, t_1, \ldots, t_{N-1}, t_N = \vartheta$, where $\Delta_i = t_{i+1} - t_i = \text{const} > 0$. We also assume

$$X_{i+1} = X(t_{i+1}; t_i, X_i),\quad i = 0,1,\ldots,N-1 \tag{2.1}$$

whence it follows that $X_N = X(\vartheta; t_0, X_0)$.

Thus, the RD $X_N$ of the DI (1.2) could have been calculated using only the recurrence relations (2.1), had we known how to calculate all the "intermediate" RDs $X_{i+1}$ of (2.1) exactly. However, we do not know how to do so. We will use the recurrence relations (2.1) in approximate calculations of the set $X_N$.

The first step of our approximate calculations corresponds to the interval $I_1 = [t_0, t_1]$ of the partition $\Gamma$.

Let $x[t_0] = x_0$ be an arbitrary point in the bounded subset $X_0$ of $R^m$. Let us consider the set $\tilde{X}_1^0 = \tilde{X}(t_1; t_0, x[t_0]) = x[t_0] + \Delta_0 F_0$, where $F_0 = F(t_0, x[t_0])$. The set $\tilde{X}_1^0$ is a RD of the DI

$$\dot{x}(t) \in F_0,\quad x[t_0] = x_0 \tag{2.2}$$

and $\tilde{X}_1^0$ is a certain approximation to the RD $X_1^0 = X(t_1; t_0, x[t_0])$ of (1.2), which is easier to compute than $X_1^0$.

We shall show that the Hausdorff distance $d(X_1^0, \tilde{X}_1^0)$ is at least an order of magnitude smaller than the increment $\Delta_0 = t_1 - t_0 > 0$. The following inequality holds

$$d(X_1^0,\ \tilde{X}_1^0) \le \omega(\Delta_0),\quad \omega(\Delta) = \Delta\omega^*((1+K)\Delta),\quad \Delta > 0 \tag{2.3}$$

where $\omega^*(\delta)$ is some positive function of $\delta > 0$ which tends to zero as $\delta \to 0$; the quantity $K$ was defined in Section 1; $\omega^*(\delta)$ and $K$ are independent of the choice of $t_0$, $t_1$, $x[t_0]$.

The proof of the inequality is as follows.

Let $x^{(1)} \in X_1^0$. Then a solution $x[t_0]$, $t \in I_1$, of the DI (1.2) with initial value $x[t_0]$ exists such that $x[t_0] = x^{(1)}$. This solution admits of the representation

$$x[t] = x[t_0] + \int_{t_0}^{t} f[\tau]d\tau, f[\tau] \in F(\tau, x[\tau]) \text{ a.e. on } I_1 \tag{2.4}$$

By Condition 1, a function $\omega^*(\delta)$ ($\omega^*(\delta) \downarrow 0$ as $\delta \downarrow 0$) exists such that

$$d(F(t_*,\ x_*),\ F(t^*,\ x^*)) \le \omega^*(|t_* - t^*| + \| x_* - x^* \|),\ d(F(t,\ x_*),\ F(t,\ x^*)) \le L\| x_* - x^* \| \tag{2.5}$$

$$(t_*, x_*), (t^*, x^*) \text{ in } D, t \in I_1$$

It follows from (2.4) and (2.5) that

$$d(F(t,\ x[t]),\ F_0) \le \omega^*(\Delta_0 + \| x[t] - x[t_0] \|) \le \omega^*((1+K)\Delta_0) \tag{2.6}$$

It follows from (2.4) and (2.5) that $f[t] \in F_0 + \omega^*((1 + K)\Delta_0)B$ for a.e. $t \in I_1$. Consequently, we have

$$\frac{1}{\Delta_0}\int_{t_0}^{t_1} f[\tau]d\tau \in F_0 + \omega^*((1+K)\Delta_0)B \tag{2.7}$$

It follows from this inclusion that

$$x[t_1] = x[t_0] + \int_{t_0}^{t_1} f[\tau]d\tau \in x[t_0] + \Delta_0(F_0 + \omega^*((1+K)\Delta_0)B) = \tilde{X}_1^0 + \omega(\Delta_0)B$$

$$\omega(\delta) = \delta\omega^*((1+K)\delta),\quad \delta \in (0,\ \infty)$$

Hence we obtain

$$X_1^0 \subset \tilde{X}_1^0 + \omega(\Delta_0)B \tag{2.8}$$

We will now prove that

$$\tilde{X}_1^0 \subset X_1^0 + \omega(\Delta_0)B \tag{2.9}$$

Let $x^{(1)} \in \tilde{X}_1^0$. It is true that $x^{(1)} = x[t_0] + \Delta f^{(1)}, f^{(1)} \in F_0$. Together with the motion $x^{(1)}[t] = x[t_0] + (t - t_0)f^{(1)}$, $t \in I_1$ of the DI $\dot{x}[t] \in F_0$ with initial value $x[t_0]$, let us consider the Euler polygon $\tilde{x}^{(k)}[t]$, $t \in I_1$, defined by the relations $\tilde{x}^{(k)}[t_{j+1}^{(k)}] = \tilde{x}^{(k)}[t_j^{(k)}] + \Delta_j^{(k)} f_j^{(k)}$, where $t_j^{(k)}$, $t_{j+1}^{(k)}$ are points of the partition $\Gamma_k = \{t_0^{(k)} = t_0, t_1^{(k)}, \ldots, t_{N(k)-1}^{(k)}, t_{N(k)}^{(k)} = t_1$; $\Delta_j^{(k)} = t_{j+1}^{(k)} - t_j^{(k)} = \text{const}$, and $f_j^{(k)} \in F(t_j^{(k)}, \tilde{x}^{(k)}[t_j^{(k)}]) = F_j^k$ is the vector in the set $F_1^{(k)}$ nearest to the vector $f^{(1)}$; $\tilde{x}^{(k)}[t_0^{(k)}] = x[t_0]$.

The Euler polygon $\tilde{x}^{(k)}[t]$, $t \in I_1$ satisfies the inequality $\| \tilde{x}^{(k)}[t] - x[t_0] \| < K\Delta_0$, and so $d(F(t, \tilde{x}^{(k)}[t]), F_0) < \omega^*((1 + K)\Delta_0)$, $t \in I_1$.

Since $f^{(1)} \in F_0$, it follows that $f^{(1)} \in F(t, \tilde{x}^{(k)}[t]) + \omega^*((1 + K)\Delta_0)B$, $t \in I_1$. In particular, the following inclusion holds for $f^{(1)}$ at the nodal points of the partition $\Gamma_k$

$$f^{(1)} \in F_j^{(k)} + \omega^*((1+K)\Delta_0)B \tag{2.10}$$

Thus, taking (2.10) into consideration, we obtain an upper limit

$$\left\| x^{(1)}[t_{j+1}^{(k)}] - \tilde{x}^{(k)}[t_{j+1}^{(k)}] \right\| \le \left\| x^{(1)}[t_j^{(k)}] - \tilde{x}^{(k)}[t_j^{(k)}] \right\| + \Delta_j^{(k)}\omega^*((1+K)\Delta_0),\quad j = 0,\ 1, \ldots,\ N(k) - 1$$

Taking these $N(k)$ inequalities into consideration, we obtain $\| x^{(1)}[t^{(k)}_{N(k)}] - \tilde{x}[t^{(k)}_{N(k)}] \| < \Delta^{(k)}_0 \omega^* ((1 + K)\Delta_0) + \Delta^{(k)}_1 \omega^* ((1 + K)\Delta_0) + \ldots + \Delta^{(k)}_{N(k)-1} \omega^* ((1 + K)\Delta_0)$.

This inequality may be rewritten in the form

$$\left\| x^{(1)}[t_1] - \tilde{x}^{(k)}[t_1] \right\| \leq \omega(\Delta_0) \tag{2.11}$$

Inequality (2.11) holds for all polygonal lines $\tilde{x}^{(k)}[t]$, $t \in I_1$ constructed in this way.

Extracting from the sequence $\tilde{x}^{(k)}[t]$, $t \in I_1 (k = 1, 2, \ldots)$ a uniformly convergent subsequence $\tilde{x}^{(n)}[t]$, $t \in I_1$ $(n = 1, 2, \ldots)$, we conclude that the vector function $x[t] = \lim_{n \to \infty} \tilde{x}^{(n)}[t]$, $t \in I_1$ satisfies the relations $\dot{x}[t] \in F(t, x[t])$, $t \in I_1$. We have thus established the inclusion $x^{(1)} = x^{(1)}[t_1] \in X_1^0 + \omega(\Delta_0)$, proving (2.9). The required relation (2.3) now follows from (2.8) and (2.9).

The RDs $X_1$ and $\tilde{X}_1$ may be expressed as

$$X_1 = \bigcup_{x[t_0] \in X_0} X(t_1, t_0, x[t_0]), \; \tilde{X}_1 = \bigcup_{x[t_0] \in X_0} \tilde{X}(t_1, t_0, x[t_0]) \tag{2.12}$$

It follows from (2.3) and (2.12) that

$$d(X_1, \tilde{X}_1) \leq \omega(\Delta_0) \tag{2.13}$$

The RD $\tilde{X}_1$ is somewhat simpler to compute than $X_1$, since it is expressible as a union of convex sets $\tilde{X}(t_1, t_0, x_0)$, $x_0 \in X_0$.

**3.** We will now describe an approximation scheme for computing RDs of system (1.1).

Divide the space $R^m$ into $m$-dimensional cubes $\Phi_j$ with centres $x_j^0 = x_j[t_0]$ and vertices distant from the centres by an amount $\varepsilon > 0$. We will call the infinite set of centres $x_j^0$ a lattice in $R^m$ and denote it by $N_\varepsilon$.

Now pick out all cubes $\Phi_j$ such that $\Phi_j \cap X_0 \neq \varnothing$, say $\Phi_j$ $(j = 1, 2, \ldots, J_0)$. Consider the centres $x_j^0$ $(j = 1, 2, \ldots, J_0)$ of these cubes. Since $X_0$ is a bounded set, $J_0$ is a finite number.

Put $X_0^\varepsilon = \{x_j^0 : j = 1, 2, \ldots, J_0\}$. By the construction of the set $X_0^\varepsilon$, we have $d(X_0, X_0^\varepsilon) < \varepsilon$.

Let $\delta > 0$ be given; starting from the set $F_0$, we define, by some rule, a finite $\delta$-net $F_0^\delta = F^\delta(t_0, x[t_0]) = \{f_k^\delta \in F_0 : k = 1, 2, \ldots, K_0\}$ such that $d(F_0, F_0{}^\delta) < \delta$.

We thus have discrete finite approximations $X_0^\varepsilon$ and $F_0^\delta$ of the sets $X_0$ and $F_0$.

Now consider the set $\tilde{X}_1$. Let $x[t_1]$ be an arbitrary point of $\tilde{X}_1$. We can represent this point in the form $x[t_1] = x[t_0] + \Delta_0 f$, $x[t_0] \in X_0$, $f \in F_0$.

We now find an approximation to the point $x[t_1]$ in the discrete scheme. In that connection, given $x[t_0]$, we find a point $x_j^0 \in X_0^\varepsilon$ distant from $x[t_0]$ by at most $\varepsilon$

$$x_j^0 \in X_0^\varepsilon, \; \left\| x_j^0 - x[t_0] \right\| \leq \varepsilon$$

Now, for a vector $f \in F_0$, a vector $f_j^* \in F(t_0, x_j^0)$ exists which satisfies the inequality

$$\left\| f - f_j^* \right\| \leq d(F_0, \; F(t_0, \; x_j^0)) \leq L \left\| x[t_0] - x_j^0 \right\| \leq L\varepsilon \tag{3.1}$$

since $x \mapsto F(t, x)$ is a Lipschitz-continuous multivalued function with constant $L = L(D)$. Taking the vector $f_j^* \in F(t_0, x_j^0)$, we can find a vector $f_j^k \in F^\delta(t_0, x_j^0)$ such that $\| f_j^* - f_j^k \| < \delta$.

Thus for any vector $f \in F_0$ a vector $f_j^k \in F^\delta(t_0, x_j^0)$ exists such that difference between the points $\tilde{x}[t_1] = x[t_0] + \Delta_0 f$ and $x^*[t_1] = x_j^0 + \Delta_0 f_j^k$ is

$$\left\| (x[t_0] + \Delta_0 f) - (x_j^0 + \Delta_0 f_j^k) \right\| \leq \varepsilon + (L\varepsilon + \delta)\Delta_0 \tag{3.2}$$

Denote $\xi_m = \exp(2L\Delta_m)$. Setting $\delta = L\varepsilon$, we see that inequality (3.2) becomes

$$\left\| \tilde{x}[t_1] - x^*[t_1] \right\| \leq \varepsilon(1 + 2L\Delta_0) \leq \xi_0 \varepsilon \tag{3.3}$$

Thus, we finally deduce that for any point $\tilde{x}[t_1] \in \tilde{X}_1$ a point $x^*[t_1] = x^0_j + \Delta_0 f^k_j, x^0_j \in X^\varepsilon_0, f^k_j \in F^\delta(t_0, x^0_j)$ exists which satisfies inequality (3.3).

Denote the set of all such points for $j = 1, 2, \ldots, J_0, k = 1, 2, \ldots, K^j_0$ by $X^*_1 = X^*(t_1, t_0, X^\varepsilon_0)$.

We thus conclude that the Hausdorff distance between the sets $\tilde{X}_1$ and $X^*_1$ satisfies the inequality $h(\tilde{X}_1, X^*_1) < \xi_0\varepsilon$. Since $d(X_0, X^\varepsilon_0) < \varepsilon$ and $F^\delta(t_0, x^0_j) \subset F(t_0, x^0_j)$, it follows that the Hausdorff distance between $X^*_1$ and $\tilde{X}_1$ satisfies an analogous inequality. These inequalities imply that

$$d(\tilde{X}_1,\ X^*_1) \le \xi_0\varepsilon \tag{3.4}$$

It follows from estimates (2.13) and (3.4) that the Hausdorff distance between the sets $X_1$ and $X^*_1$ satisfies the inequality

$$d(X_1,\ X^*_1) \le \xi_0\varepsilon + \omega(\Delta_0) \tag{3.5}$$

For each point $x[t_1] \in X^*_1$, we find a point $x^1_j = x_j[t_1]$ of the lattice $N_\varepsilon$ distant from $x[t_1]$ by at most $\varepsilon : \| x^1_j - x[t_1] \| < \varepsilon$. Combining all such points $x^1_j$ of $N_\varepsilon$ corresponding to all points $x[t_1]$ of $X^*_1$ into a single set $X^\varepsilon_1$, we infer from (3.5) that

$$d(X_1,\ X^\varepsilon_1) \le d(X_1,\ X^*_1) + d(X^*_1,\ X^\varepsilon_1) \le \xi_0\varepsilon + \omega(\Delta_0) + \varepsilon$$

The discrete finite set $X^\varepsilon_1$ thus constructed is an $\varepsilon_1$-net for the set $X_1$—a reachable domain of the DI (1.2), where $\varepsilon_1 = \xi_0\varepsilon + \omega(\Delta_0) + \varepsilon$.

We will now construct approximations in the next interval $I_2[t_1, t_2]$ of the partition $\Gamma$. The arguments are entirely analogous to those above, with very slight differences.

Consider the RD $X_1$, Let $x_1 = x[t_1] \in X_1$. Let us consider the set $\tilde{X}^1_2 = \tilde{X}(t_2,\ t_1,\ x[t_1]) = x[t_1] + \Delta_1 F_1$, $\Delta_1 = t_2 - t_1$, which is a RD of the DI $\dot{x}(t) \in F_1, x[t_1] = x_1$ for a time $t_2$, where $F_1 = F(t_1, x[t_1])$. $\tilde{X}^1_2$ approximates the RD $X^1_2 = X(t_2; t_1, x[t_1])$, and, by analogy with (2.3), the Hausdorff distance between these two sets satisfies the inequality $d(X^1_2, \tilde{X}^1_2) < \omega(\Delta_1)$. Consequently, the Hausdorff distance between the sets $X_2$ and $\tilde{X}_2 = \tilde{X}(t_2; t_1, X_1) = \cup_{x[t_1]\in X_1} X^1_2(t_2; t_1, x[t_1])$ satisfies the inequality

$$d(X_2,\ \tilde{X}_2) \le \omega(\Delta_1) \tag{3.6}$$

Consider the set $\tilde{X}_2$. Let $\tilde{x}[t_2] \in \tilde{X}_2$ be an arbitrary point. Then it may be expressed as $\tilde{x}[t_2] = x[t_1] + \Delta_1 f_1$, where $x[t_1] \in F_1$.

For the point $x[t_1] \in X_1$, find a point $x^1_j$ of the $\varepsilon_1$-net $X^\varepsilon_1$ of $X_1$ such that $\| x^1_j - x[t_1] \| < \varepsilon_1$. Given a number $\delta > 0$ and using some rule, we define on the set $F_1$ a finite $\delta$-net

$$F^\delta_1 = F^\delta(t_1,\ x[t_1]) = \{f^\delta_k \in F_1 : k = 1,\ 2, \ldots,\ K_1\}$$

such that $d(F_1, F_1{}^\delta) < \delta$.

Assuming that the $\delta$-net $F^\delta(t_1, x^1_j)$ is given, by analogy with (3.1) and (3.2) we can find, for any vector $f \in F_1$, a vector $f^k_j \in F^\delta(t_1, x^1_j)$ such that the distance between the points $\tilde{x}[t_2] = x[t_1] + \Delta_1 f$ and $x^*[t_2] = x^1_j + \Delta_1 f^k_j$ is

$$\left\|(x[t_1] + \Delta_1 f) - (x^1_j + \Delta_1 f^k_j)\right\| \le \left\|x[t_1] - x^1_j\right\| + \Delta_1\left\|f - f^k_j\right\| \le \varepsilon_1 + \Delta_1(L\varepsilon_1 + \delta)$$

Letting $\delta > 0$ be any number such that $\delta < L\varepsilon_1$ (for example, $\delta = L\varepsilon_1$), we obtain the estimate

$$\left\|\tilde{x}[t_2] - x^*[t_2]\right\| \le \varepsilon_1(1 + 2L\Delta_1) \le \xi_1\varepsilon_1 \tag{3.7}$$

Thus, we finally deduce that for any point $\tilde{x}[t_2] \in \tilde{X}_2$ there is a point

$$x^*[t_2] = x^1_j + \Delta_1 f^k_j,\quad (x^1_j \in X^\varepsilon_1,\ f^k_j \in F^\delta(t_1,\ x^1_j),\quad 0 < \delta \le L\varepsilon_1)$$

satisfying inequality (3.7).

Denote the set of all points $x^*[t_2] = x_j^1 + \Delta_1 f_j^k, x_j^1 \in X_1^\varepsilon, f_j^k \in F^\delta(t_1, x_j^1); j = 1, 2, \ldots, J_1, k = 1, 2, \ldots, K_1^j$ by $X_2^* = X^*(t_2, x_1, X_1^\varepsilon)$.

Inequality (3.7) implies that the Hausdorff distance between the sets $\tilde{X}_2$ and $X_2^*$ satisfies the inequality $h(X_2, X_2^*) < \xi_1\varepsilon_1$. On the other hand, noting that $d(X_1, X_1^\varepsilon) < \varepsilon_1$ and $F^\delta(t_1, x^1_j) \subset F(t_1, x^1_j)$, we obtain $h(X_2^*, \tilde{X}_2) < \xi_1\varepsilon_1$.

Consequently

$$d(\tilde{X}_2,\ X_2^*) \le \xi_1\varepsilon_1 \tag{3.8}$$

Taking estimates (3.6) and (3.8) into account, we obtain

$$d(X_2,\ X_2^*) \le \xi_1\varepsilon_1 + \omega(\Delta_1) \tag{3.9}$$

For every point $x[t_2] \in X_2^*$, we find a point $x_j^2 \in N_\varepsilon$ distant by not more than $\varepsilon$ from $x[t_2]$. Denote the set of all points $x_j^2$ for points $x[t_2] \in X_2^*$ by $X_2^\varepsilon$. By construction of $X_2^\varepsilon$, we have an estimate $d(X_2^*, X_2^\varepsilon) < \varepsilon$. Taking note of (3.9) and the last inequality, we obtain $d(X_2, X_2^\varepsilon) < \varepsilon_2$, $\varepsilon_2 = \xi_1\varepsilon_1 + \omega(\Delta_1) + \varepsilon$.

Thus, $X_2^\varepsilon$ is a discrete approximation to the RD $X_2$.

Similar constructions yield approximating sets $X_3^\varepsilon, X_4^\varepsilon, \ldots, X_{i+1}^\varepsilon, \ldots, X_N^\varepsilon$ for the intervals $[t_2, t_3]$, $[t_3, t_4], \ldots, [t_i, t_{i+1}], \ldots, [t_{N-1}, t_N]$. In the interval $[t_j, t_{j+1}]$ we have the estimate

$$d(X_{i+1},\ X_{i+1}^\varepsilon) \le \varepsilon_{i+1},\ \ \varepsilon_{i+1} = \xi_i\varepsilon_i + \omega(\Delta_i) + \varepsilon \tag{3.10}$$

for the Hausdorff distance between the RD of the DI (1.2) and the computed discrete approximation. In particular, in the last interval $[t_{N-1}, t_N]$ we have $d(X_N, X_N^\varepsilon) < \varepsilon_N$.

Using the recurrent formula (3.10) for the sequence $\{\varepsilon_i\}$, as well as the expressions $\omega(\Delta) = \Delta\omega^*((1 + K)\Delta)$, where $\omega^*((1 + K)\Delta \downarrow 0$ as $\Delta \downarrow 0$, and the equalities $\Delta_0 + \Delta_1 \ldots + \Delta_{N-1} = \vartheta - t_0$, $\Delta_i = \Delta$), we obtain an upper bound for $\varepsilon_N$

$$\varepsilon_N \le \exp[2L(\vartheta - t_0)](\vartheta - t_0)\{(1 + 1/N)\Omega\sqrt{\Delta} + \omega^*((1+K)\Delta)\} \tag{3.11}$$

(the number $\varepsilon$, the parameter of the lattice $N_\varepsilon$ is related to the length $\Delta = (\vartheta - t_0)/N$ of the intervals in $\Gamma$ by a formula $\varepsilon = \Omega\Delta\sqrt{\Delta}$, where $\Omega$ is some finite positive number).

It follows from estimate (3.11) that its right-hand side tends to zero as $\Delta \to 0$. At the same time, $d(X_N, X_N^\varepsilon) \to 0$ as $\Delta \to 0$. The rate of convergence is determined by the expression in braces on the rights of (3.11). As $\Delta \to 0$ the number $N$ tends to infinity, so that for large $N$

$$(1 + 1/N)\Omega\sqrt{\Delta} + \omega^*((1+K)\Delta) \approx \Omega\sqrt{\Delta} + \omega^*((1+K)\Delta) \tag{3.12}$$

When system (1.1) is autonomous (i.e. the right-hand side of the system has the form $f(x, u)$), we may assume, in addition to conditions 1 and 2, that the expression on the right of the first inequality in (2.5) is equal to $L\|x^* - x_*\|$. In that case (3.12) becomes

$$(1 + 1/N)\Omega\sqrt{\Delta} + LK\Delta \approx \Omega\sqrt{\Delta} + LK\Delta$$

*Example.* Suppose that the dynamics of a control system are described by the equations

$$\dot{x}_1 = \frac{1}{2}x_1(1 - x_2) + u_1,\ \ \dot{x}_2 = \frac{1}{2}x_2(1 - x_1) + u_2,\ \ |u_1| \le 1,\ |u_2| \le 1$$

For

$$t_0 = 0,\ X_0 = \{(x_1,\ x_2) \in R^2 : (x_1 - 5)^2 + (x_2 - 5)^2 \le 25\},\ \ \Delta = 0.1,\ \varepsilon = 0.025$$

the RDs of the system corresponding to times $t_0 = 0$, $t_3 = 0.3$, $t_{14} = 1.4$ are shown in Fig. 1.

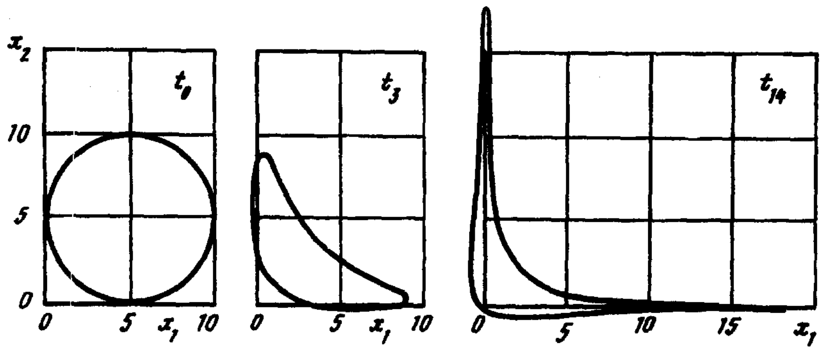


Fig. 1.

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